Reasoning in the real-world with BL Multi-Modal Logic

Topology, Algebra, and Categories in Logic (TACL) 2026

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July 28th, 2026

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Table of contents

Temporal and spatial logics

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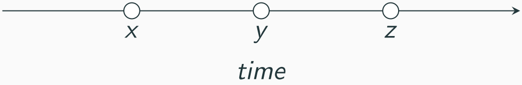
Let's consider, for example, some ordered **points** in time ($x < y < z$).

We can define **relations** between points in time as follows:

$$\begin{aligned} F(x, y) &: x < y && \textit{future} \\ P(x, y) &: y < x && \textit{past} \\ =(x, y) &: x = y && \textit{equality} \end{aligned}$$



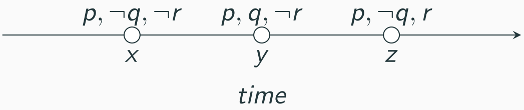
Temporal and spatial logics

Temporal and **spatial** logics are critical in modeling **real-world** scenarios.

Let's consider, for example, some ordered **points** in time ($x < y < z$).

We can associate **events** with points in time:

- p: I am doing a PhD
- q: I am giving this talk
- r: I am having dinner



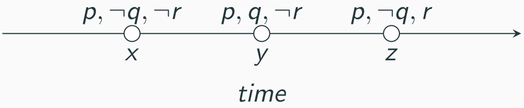
Temporal and spatial logics

Temporal and **spatial** logics are critical in modeling **real-world** scenarios.

Let's consider, for example, some ordered **points** in time ($x < y < z$).

We can **reason** about events in time:

- $q \wedge r$ (is there a point in time where I am both giving this talk and getting dinner?) ✗
- $q \wedge \langle F \rangle r$ (is there a point in time where I am giving this talk and in the future I am getting dinner?) ✓

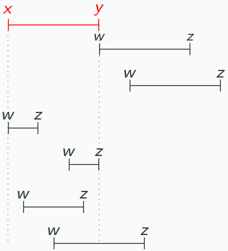


Temporal and spatial logics

Temporal and **spatial** logics are critical in modeling **real-world** scenarios.

Let's consider, for example, time **intervals** $I(x, y) : x < y$.

We can define **relations** between time intervals:

relation	**definition**	**example**
after	$R_A([x, y], [w, z]) \quad = \quad = (y, w)$	
later	$R_L([x, y], [w, z]) \quad = \quad < (y, w)$	
begins	$R_B([x, y], [w, z]) \quad = \quad = (x, w) \wedge < (z, y)$	
ends	$R_E([x, y], [w, z]) \quad = \quad < (x, w) \wedge = (y, z)$	
during	$R_D([x, y], [w, z]) \quad = \quad < (x, w) \wedge < (z, y)$	
overlaps	$R_O([x, y], [w, z]) \quad = \quad < (x, w) \wedge < (w, y) \wedge < (y, z)$	

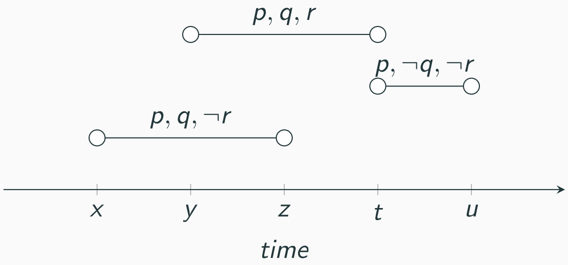
Temporal and spatial logics

Temporal and **spatial** logics are critical in modeling **real-world** scenarios.

Let's consider, for example, time **intervals** $I(x, y) : x < y$.

We can associate **events** with time intervals and **reason** about them:

- $q \wedge \langle O \rangle r$ (is there an interval of time in which I am giving this talk overlapping with another in which I am getting dinner?) ✓

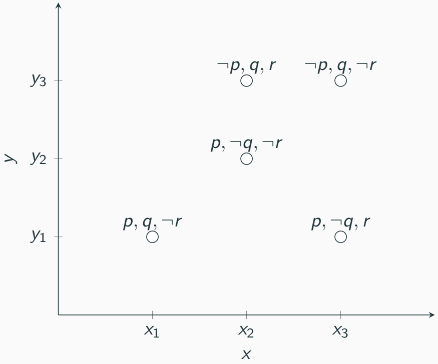


Temporal and spatial logics

Temporal and **spatial** logics are critical in modeling **real-world** scenarios.

Let's consider, for example, **points** in space $P(x, y)$.

We can define **relations** between points in space (*up, right, down, left*), associate **events** with those points, and **reason** about them:

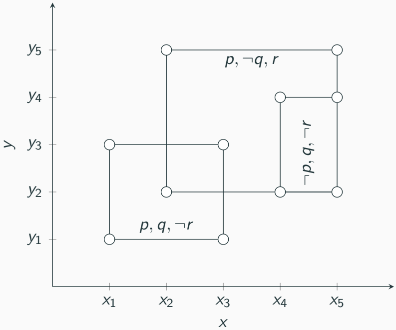


Temporal and spatial logics

Temporal and **spatial** logics are critical in modeling **real-world** scenarios.

Let's consider, for example, **rectangles** $R(I_x, I_y)$, where I_x, I_y are intervals associated to axis x and y respectively.

We can define **relations** between rectangles (*overlapping, disconnected,* $\ldots$), associate **events** with those rectangles, and **reason** about them:



A general framework for temporal and spatial logics

Goal: logical framework for uniform and general treatment of temporal and spatial logics (e.g., fuzzification, reasoning, symbolic learning):

- linear temporal logic
- interval temporal logic
- pointwise spatial logic
- spatial logic of topological relations

Solution: Multi-Modal Logic K_n

Multi-Modal Logic

Multi-Modal Logic (syntax)

Definition

Let $\mathcal{A}P$ be a set of propositional letters, $\neg$ and $\vee$ the classical Boolean connectives, and $\{\langle X_1 \rangle, \ldots, \langle X_n \rangle\}$ a finite set of existential modalities. Well-formed *Multi-Modal Logic K_n* formulas are obtained as follows:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle X_i \rangle \varphi,$$

for $1 \leq i \leq n$ and $p \in \mathcal{A}P$.

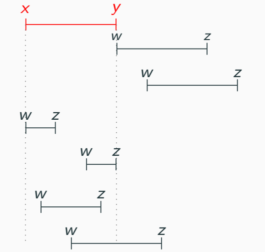
$\wedge$, $\rightarrow$, and $[X_i]$ are derivable in the usual way (e.g., $[X_i]\varphi \equiv \neg\langle X_i \rangle \neg\varphi$).

Example: Halpern and Shoham's Modal logic of Time Intervals

Let $\mathbb{D} = \langle D, < \rangle$ be a linear order with domain D.

An interval over $\mathbb{D}$ is an ordered pair $[x, y]$, where $x, y \in \mathbb{D}$ and $x < y$.

There are 12 different binary ordering relations between two intervals:

relation	**definition**	**example**
after	$R_A([x,y],[w,z]) \quad = \quad =(y,w)$	
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and their inverse $R_{\overline{X}} = R_X^{-1}$ for each $X \in \{A, L, B, E, D, O\}$.

To each relation $R_{X \in \{A,\overline{A},L,\overline{L},B,\overline{B},E,\overline{E},D,\overline{D},O,\overline{O}\}}$ corresponds a modality $\langle X \rangle$.

Kripke frames

Definition

Given a non-empty set of *worlds* W, a *Kripke frame* is an object $F = \langle W, R_1 \dots R_n \rangle$ where each $R_i \subseteq W \times W$ is an *accessibility* relation.

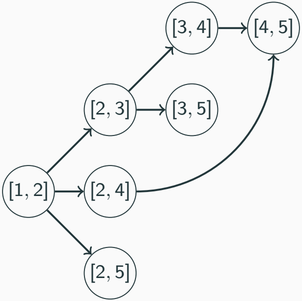

Figure 1: A Kripke frame for the relation R_A (*after*) of Halpern and Shoham's Modal Logic of Time Intervals; each world w_i represents an interval $[x_i, y_i]$.

Kripke structures

Definition

A *Kripke structure* (or *model*) is a Kripke frame enriched with a *valuation function* $V : W \to 2^{\mathcal{AP}}$, and it is denoted by $M = \langle F, V \rangle$.

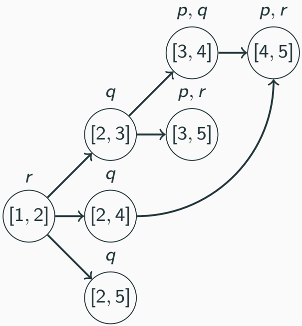

Figure 2: A Kripke structure for the Kripke frame in Fig. 1 and the set of propositional letters $\mathcal{AP} = \{p, q, r\}$; for each world, we represent only the propositional letters which are true in that world.

Multi-Modal Logic (semantics)

Definition

Given a well-formed formula φ, we say that φ is *satisfied in M at w*, for some world w, and we denote it by $M, w \Vdash \varphi$, if and only if

$$
\begin{array}{lll}
M, w \Vdash p & \text{iff} & w \in V(p), \text{ for each } p \in \mathcal{A}P, \\
M, w \Vdash \neg\psi & \text{iff} & M, w \nVdash \psi, \\
M, w \Vdash \psi \vee \xi & \text{iff} & M, w \Vdash \psi \text{ or } M, w \Vdash \xi \\
M, w \Vdash \langle X_i \rangle \psi & \text{iff} & \text{there is } s \text{ s.t. } wR_i s \text{ and } M, s \Vdash \psi.
\end{array}
$$

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\end{array}
$$

Definition

A formula φ is *satisfiable* iff there exists a structure and a world in which it is satisfied, and *valid* if it is satisfied at every world in every structure.

A general framework for temporal and spatial logics

Objective: leverage a uniform and general framework, in this case **Multi-Modal Logic** K_n, to define (meta-)algorithms parametrized:

- on the number of **linear orders**
- on the type of **events** (worlds) – points, intervals, rectangles, ...
- on the **relations** between such events – future, past, overlapping, ...

Such framework is suitable for uniform **fuzzification**, **reasoning** and **(symbolic) learning**.

Basic Fuzzy Logic (BL)

Motivation

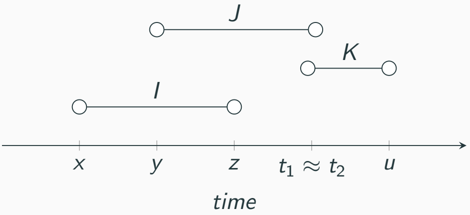
Common challenges in temporal and spatial data:

- **Inacuraccies** in the data
  - sensing process
  - discretization process
  - digitization process
  - ...
- **Unclear boundaries** due to treating numerical values semantically
  - When is a temperature high/low?
  - When can a person be considered tall?
  - When can a person be considered rich?

A simple example

Example: consider we are sensing data with photocells, and, analysing their measurements, points t_1 and t_2 appear really close to each other.

What if in real life they were the same point? What if one came after the other? (and in that case, which came first?)



Solution: consider all 3 possibilities (possibly, with different values)

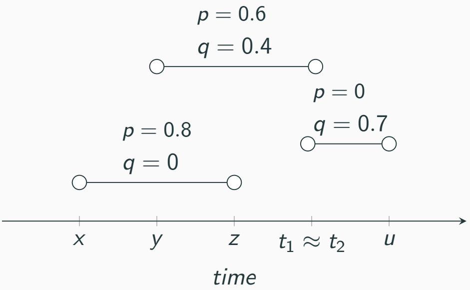
E.g., $overlaps(J, K) = 0.7, \quad after(J, K) = 0.5, \quad later(J, K) = 0.2$

A simple example

And, as we said, events can also be **partially true**.

E.g., consider two photocells also tell the probability of a **false positive**:

- p: 1^{st} photocell actually saw something (avg. accuracy on interval)
- q: 2^{nd} photocell actually saw something (avg. accuracy on interval)



Hypothesis: maybe something is moving from one area to the other?

(Continuous) t-norms

Definition

A t-norm is a binary operation $*$ on $[0,1]$ satisfying the following:

(i) $*$ is commutative and associative, i.e., for all $x, y, z \in [0,1]$,

$$\begin{aligned} x * y &= y * x \\ (x * y) * z &= x * (y * z) \end{aligned}$$

(ii) $*$ is non-decreasing in both arguments, i.e.

$$\begin{aligned} x_1 \leq x_2 \quad &\text{implies} \quad x_1 * y \leq x_2 * y \\ y_1 \leq y_2 \quad &\text{implies} \quad x * y_1 \leq x * y_2 \end{aligned}$$

(iii) $1 * x = x$ and $0 * x = 0$ for all $x \in [0,1]$.

$*$ is a *continuous t-norm* if it is a continuous mapping of $[0,1]^2$ into $[0,1]$.

Examples of continuous t-norms

The most important examples of continuous t-norms are

(i) Łukasiewicz t-norm:

$$x * y = max(0, x + y - 1)$$

(ii) Gödel t-norm:

$$x * y = min(x, y)$$

(iii) Product t-norm:

$$x * y = x \cdot y \quad \text{(product of reals)}$$

A general definition of implication

Definition

Let $*$ be a continuous t-norm. Then there is a unique operation

$$x \Rightarrow y$$

satisfying, for all $x, y, z \in [0, 1]$, the condition

$$(x * z) \leq y \quad \text{iff} \quad z \leq (z \Rightarrow y),$$

namely,

$$x \Rightarrow y = max\{z|x * z \leq y\}.$$

The operation $x \Rightarrow y$ is called the *residuum* of the t-norm.

Examples of implications

The following operations are residua of the three examples of t-norms.

For $x \leq y, x \Rightarrow y = 1$. For $x > y$,

(i) Łukasiewicz implication:

$$x \Rightarrow y = \ y - x$$

(ii) Gödel implication:

$$x \Rightarrow y = y$$

(iii) Goguen implication:

$$x \Rightarrow y = y/x \quad \text{(residuum of product conjunction)}$$

A definition of BL-algebra

Definition

A BL-algebra is an algebra

$$(L, \cap, \cup, *, \Rightarrow, 0, 1)$$

with four binary operations and two constants such that

1. $(L, \cap, \cup, 0, 1)$ is a lattice with the largest element 1 and the least element 0 (with respect to the lattice ordering $\leq$)
2. $(L, *, 1)$ is a commutative semigroup with the unit element 1, i.e., $*$ is commutative, associative, $1 * x = x$ for all x
3. $*$ and $\Rightarrow$ form an adjoint pair, i.e., $z \leq (x \Rightarrow y)$ iff $x * z \leq y$ for all x, y, z
4. $x \cap y = x * (x \Rightarrow y)$ for all $x, y \in L$
5. $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$ for all $x, y \in L$ (prelinearity)

BL Multi-Modal Logic

BL Multi-Modal Logic (syntax)

Definition

Let $\mathcal{A}P$ be a set of propositional letters, $\mathcal{A}$ a BL-algebra. Well-formed formulas of *Multi-Modal Logic BL-K_n* are obtained as:

$$\varphi ::= \alpha \mid p \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \langle X_i \rangle \varphi \mid [X_i]\varphi,$$

for $1 \leq i \leq n$, $p \in \mathcal{A}P$, and $\alpha \in \mathcal{A}$.

Problem: the double negation axiom ($\neg\neg\varphi \equiv \varphi$) is **not** always vaild.

Solution: explicit inclusion of all Boolean operators, as well as the universal version of every modality.

BL-Kripke frames and structures

Definition

Given a non-empty set of *worlds* W and a BL-algebra $\mathcal{A}$, an *BL-Kripke frame* is an object

$$\widetilde{F} = \langle W, \widetilde{R}_1 \ldots, \widetilde{R}_n \rangle,$$

where each $\widetilde{R}_i : (W \times W) \to \mathcal{A}$ is an *accessibility* function.

BL-Kripke frames and structures

Definition

Given a non-empty set of *worlds* W and a BL-algebra $\mathcal{A}$, an *BL-Kripke frame* is an object

$$\widetilde{F} = \langle W, \widetilde{R}_1 \ldots, \widetilde{R}_n \rangle,$$

where each $\widetilde{R}_i : (W \times W) \to \mathcal{A}$ is an *accessibility* function.

Definition

An *BL-Kripke structure* (or *model*) is a BL-Kripke frame enriched with a *valuation function* $\widetilde{V} : (W \times \mathcal{A}P) \to \mathcal{A}$, and it is denoted by

$$\widetilde{M} = \langle \widetilde{F}, \widetilde{V} \rangle.$$

BL Multi-Modal Logic (semantics)

Definition

Given a well-formed formula φ, we compute its *value in* $\widetilde{M}$ *at* w, for some $w \in W$, by extending $\widetilde{V}$ to formulas, as follows:

$$
\begin{array}{lcl}
\widetilde{V}(\alpha, w) & = & \alpha, \\
\widetilde{V}(\varphi \wedge \psi, w) & = & \widetilde{V}(\varphi, w) * \widetilde{V}(\psi, w), \\
\widetilde{V}(\varphi \vee \psi, w) & = & \widetilde{V}(\varphi, w) \cup \widetilde{V}(\psi, w), \\
\widetilde{V}(\varphi \rightarrow \psi, w) & = & \widetilde{V}(\varphi, w) \Rightarrow \widetilde{V}(\psi, w), \\
\widetilde{V}(\langle X_i \rangle \varphi, w) & = & \bigcup \{\widetilde{R}_i(w, s) * \widetilde{V}(\varphi, s)\}, \\
\widetilde{V}([X_i]\varphi, w) & = & \bigcap \{\widetilde{R}_i(w, s) \Rightarrow \widetilde{V}(\varphi, s)\}.
\end{array}
$$

The α-satisfiability problem

Definition

A formula φ of BL-K$_n$ is α-*satisfied* at world w in a BL-Kripke structure $\widetilde{M}$ if and only if

$$\widetilde{V}(\varphi, w) \geq \alpha.$$

The α-satisfiability problem

Definition

A formula φ of BL-K$_n$ is *α-satisfied* at world w in a BL-Kripke structure $\widetilde{M}$ if and only if

$$\widetilde{V}(\varphi, w) \geq \alpha.$$

Definition

A formula φ is *α-satisfiable* if and only if there exists a structure and a world in which it is α-satisfied, and it is *satisfiable* if it is α-satisfiable for some $\alpha \in \mathcal{A}$, $\alpha > 0$; respectively, a formula is *α-valid* if it is α-satisfied at every world in every model, and *valid* if it is 1-valid.

Example: Halpern and Shoham's Modal Logic of Time Intervals

Definition

Let $\mathcal{A} = \langle A, \cap, \cup, *, \Rightarrow, 0, 1 \rangle$ a complete BL-algebra.

An *BL-linear order* is a structure of the type

$$\widetilde{\mathbb{D}} = \langle D, \widetilde{<}, \widetilde{=} \rangle,$$

where D is a *domain* enriched with two functions $\widetilde{<}, \widetilde{=} \colon D \times D \to A$, for which the following conditions apply for every $x, y, z \in D$:

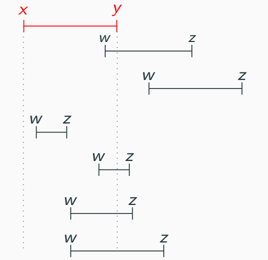
$$\begin{array}{l}
\widetilde{=}(x, y) = 1 \text{ iff } x = y, \\
\widetilde{=}(x, y) = \widetilde{=}(y, x), \\
\widetilde{<}(x, x) = 0, \\
\widetilde{<}(x, z) \geq \widetilde{<}(x, y) * \widetilde{<}(y, z), \\
\text{if } \widetilde{<}(x, y) > 0 \text{ and } \widetilde{<}(y, z) > 0, \text{ then } \widetilde{<}(x, z) > 0, \\
\text{if } \widetilde{<}(x, y) = 0 \text{ and } \widetilde{<}(y, x) = 0, \text{ then } \widetilde{=}(x, y) = 1, \\
\text{if } \widetilde{=}(x, y) > 0, \text{ then } \widetilde{<}(x, y) < 1.
\end{array}$$

Example: Halpern and Shoham's Modal Logic of Time Intervals

Let $\widetilde{\mathbb{D}} = \langle D, \widetilde{<}, \widetilde{=} \rangle$ be a BL-linear order with domain D.

An interval over $\widetilde{\mathbb{D}}$ is an ordered pair $[x, y]$, where $x, y \in \mathbb{D}$ and $x \widetilde{<} y$.

There are 12 different binary ordering relations between two intervals:

relation	definition	example
after	$\widetilde{R}_A([x,y],[w,z]) \quad = \quad \widetilde{=}(y,w)$	
later	$\widetilde{R}_L([x,y],[w,z]) \quad = \quad \widetilde{<}(y,w)$	
begins	$\widetilde{R}_B([x,y],[w,z]) \quad = \quad \widetilde{=}(x,w) * \widetilde{<}(z,y)$	
ends	$\widetilde{R}_E([x,y],[w,z]) \quad = \quad \widetilde{<}(x,w) * \widetilde{=}(y,z)$	
during	$\widetilde{R}_D([x,y],[w,z]) \quad = \quad \widetilde{<}(x,w) * \widetilde{<}(z,y)$	
overlaps	$\widetilde{R}_O([x,y],[w,z]) \quad = \quad \widetilde{<}(x,w) * \widetilde{<}(w,y) * \widetilde{<}(y,z)$	

and their inverse $R_{\overline{X}} = R_X^{-1}$ for each $X \in \{A, L, B, E, D, O\}$.

To each relation $R_{X \in \{A, \overline{A}, L, \overline{L}, B, \overline{B}, E, \overline{E}, D, \overline{D}, O, \overline{O}\}}$ corresponds a modality $\langle X \rangle$.

Analytic Tableau for BL Multi-Modal Logic

Overview of the tableau structure

- Tree-like structure with nodes and branches
- Each node is associated with a **decoration**

$$Q(\alpha \leq \varphi, w, \widetilde{F}) \text{ or } Q(\varphi \leq \alpha, w, \widetilde{F}),$$

  where Q is a judgment (either *true* or *false*), $\alpha \in A$ is a truth value from the chosen BL-algebra, φ is a formula, w is a world, and $\widetilde{F}$ is a BL-Kripke structure
- Branches represent possible **evaluations** and are associated with a BL-Kripke structure

Overview of the tableau procedure

- Systematically apply **expansion rules** to generate new nodes
- Close branches that contain contradictions using **closing rules**
- Systematically explore possible valuations to determine:
  - α-**satisfiability**, if starting from *true*$(\alpha \leq \varphi)$ it finds an open branch
  - α-**validity**, if starting from *false*$(\alpha \leq \varphi)$ it closes all branches

Example: $\Diamond(p \wedge \alpha) \wedge \alpha$ is α-satisfiable in G3

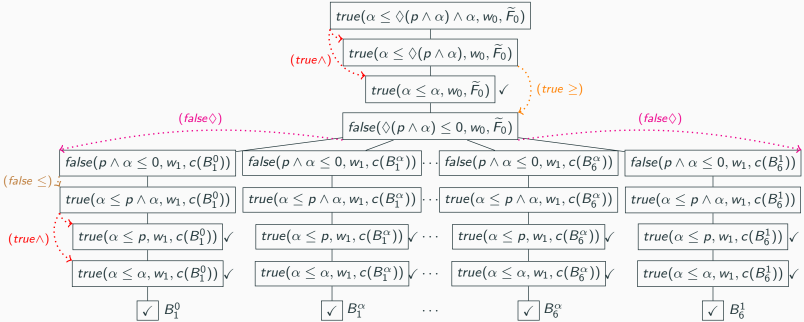


Figure 3: Fully evaluated SAT-tableau for $\alpha \leq \Diamond(p \wedge \alpha) \wedge \alpha$ in G3. Note that we only needed to fully expand one branch (e.g., B_1^0), to prove α-satisfiability; here, we choose to show all possible branches for example purposes.

Example: $\Diamond(p \wedge \alpha) \wedge \alpha$ is not α-satisfiable in Ł3

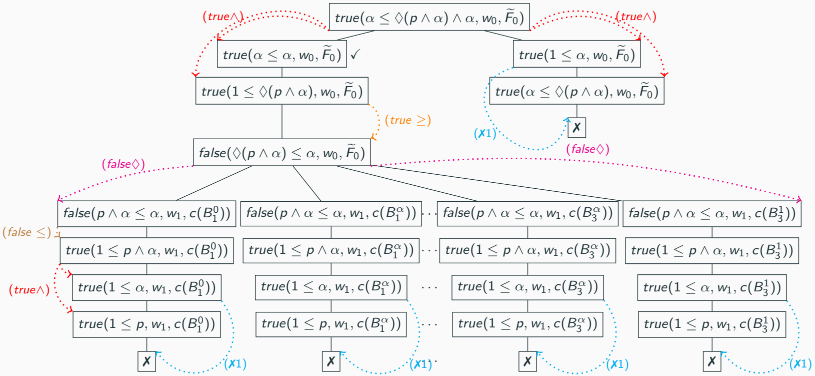


Figure 4: Fully evaluated SAT-tableau for $\alpha \leq \Diamond(p \wedge \alpha) \wedge \alpha$ in Ł3. Applying the rules of the tableau system results in closing all branches, proving that the formula is not α-satisfiable in Ł3.

Soundness and Completeness

- (**Soundness**) If a formula φ is α-satisfiable, then there exists an opened tableau for φ and α
- (**Completeness**) If a tableau is opened for φ and α, then φ is α-satisfiable (i.e., the method explores all necessary valuations)
- Similar statements can be made for α-validity

SoleReasoners.jl

The reasoning system has been implemented in the **Julia** programming language as part of **SOLE**, a framework for learning, reasoning and post hoc analysis, and is provided through the **SoleReasoners.jl** package.

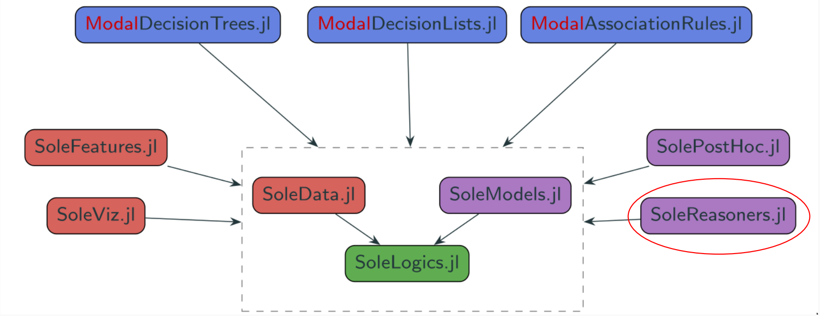


Figure 5: Symbolic Oriented Learning Environment (SOLE) ecosystem.

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Next steps: provide a reasoning system designed around BL-algebras.

Thank you for your attention!

Questions?