Modal $\mathrm{FL_{ew}}$-algebra satisfiability through first-order translation

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Introduction

Modal logics in the real world

Modal logics offer a valid treatment for **temporal** and **spatial** data, critical in modeling many **real-world** scenarios:

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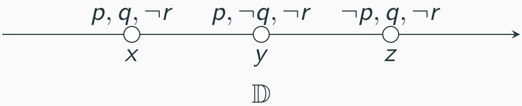

Figure 1: An example of a **point-based temporal model** with three points: $\langle F \rangle q$ holds at point x, $[P]p$ holds at point z, $\langle F \rangle \langle P \rangle q$ holds at point y.

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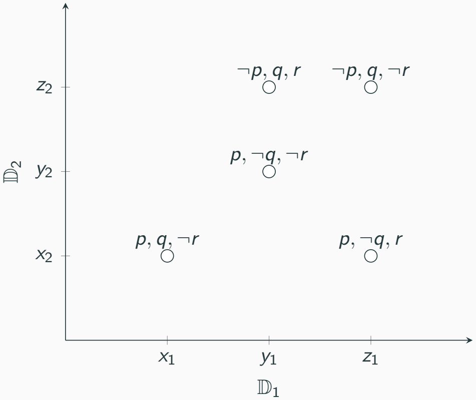

Figure 2: An example of a **point-based spatial model** with nine points. In this model, $\langle U\rangle\langle R\rangle q$ is satisfied at P_1, and $[D][L]\neg r$ at P_2.

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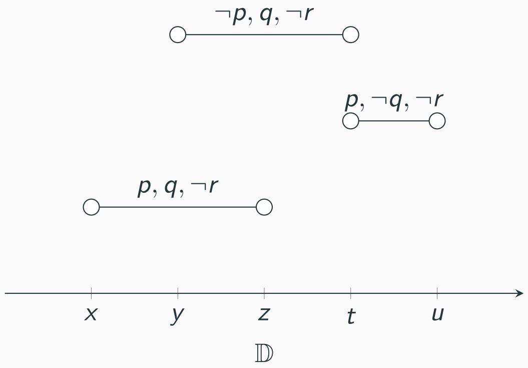

Figure 3: An **interval model** with five points and ten intervals. The intervals with a non-null valuation function are $I_1 = [x, z]$, $I_2 = [y, t]$, and $I_3 = [t, u]$. In this model, $\langle O \rangle q$ is true at I_1 and $[\overline{L}]\neg r$ is true at I_3.

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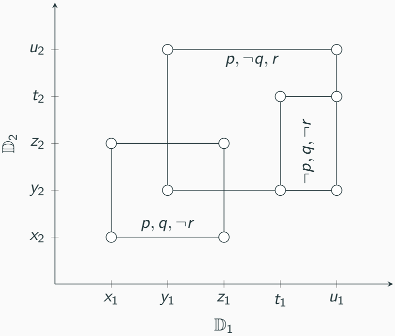

Figure 4: An example of a **topological model**. Three rectangles have non-null valuation function in it, namely $R_1 = ([x_1, z_1], [x_2, z_2])$, $R_2 = ([y_1, u_1], [y_2, u_2])$, and $R_3 = ([t_1, u_1], [y_2, t_2])$. $\langle TPP \rangle q$ is true at R_2 and $\neg [PO] q$ is true at R_1.

Challenges

Sensing and discretizing signals introduce **inaccuracies** in the data.

A common approach to deal with **uncertainty** and **unclear boundaries** in the data is through **fuzzy logics**:

- Łukasiewicz logic
- Gödel logic
- Product logic

Challenges

Sensing and discretizing signals introduce **inaccuracies** in the data.

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- Product logic

On the other hand, Melvin Fitting proposed in [1] a many-valued approach leveraging **Heyting algebras** to tackle **many-expert** scenarios.

We are interested in a framework able to generalize both problems.

[1] M. Fitting. **"Many-valued modal logics".** In: *Fundamenta Informaticae* 15.3-4 (1991), pp. 235–254

Solution: $\mathrm{FL_{ew}}$-algebras

$\mathrm{FL_{ew}}$-**algebras** encompass several known algebras:

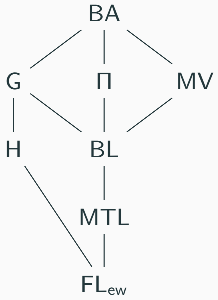

Figure 5: A partial taxonomy of well-known many-valued algebras, namely: Boolean algebra (BA), Gödel algebras (G), Product algebras (Π), MV-algebras (MV), Heyting algebras (H), Basic Fuzzy Logic algebras (BL), Monoidal t-norm logic algebras (MTL), and $\mathrm{FL_{ew}}$-algebras ($\mathrm{FL_{ew}}$).

$\mathrm{FL_{ew}}$-algebras

Definition

$\mathrm{FL_{ew}}$-**algebras**[2]

$$\mathcal{A} = \langle A, \cap, \cup, \cdot, 0, 1\rangle$$

are defined over **bounded commutative residuated lattices**, where:

- $\langle A, \cap, \cup, 0, 1\rangle$ represents a **bounded complete lattice**
- $\langle A, \cdot, 1\rangle$ is a **commutative monoid**
- We can define an **implication** $\hookrightarrow$ (the residuation property holds)

$$\alpha \hookrightarrow \beta = \sup\{\gamma \mid \alpha \cdot \gamma \preceq \beta\}$$

[2] Hiroakira Ono and Yuichi Komori. **"Logics without the contraction rule".** In: *The Journal of Symbolic Logic* 50.1 (1985), pp. 169–201

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$\mathcal{A}$ is a **chain** if $\langle A, \preceq \rangle$ is a total order, **finite** if A is finite.

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Many-valued multi-modal logics

Multi-modal logics

Definition

Let $\mathcal{A}P$ be a set of propositional letters, $\neg$ and $\vee$ the classical Boolean connectives, and $\{\langle X_1\rangle, \ldots, \langle X_n\rangle\}$ a finite set of existential modalities. Well-formed *Multi-Modal Logic* K_n[3] formulas are obtained as follows:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle X_i\rangle\varphi,$$

for $1 \leq i \leq n$ and $p \in \mathcal{A}P$.

$\wedge$, $\rightarrow$, and $[X_i]$ are derivable in the usual way (e.g., $[X_i]\varphi \equiv \neg\langle X_i\rangle\neg\varphi$).

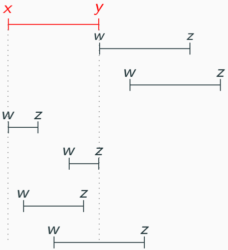
[3] P. Blackburn, M. de Rijke, and Y. Venema. **Modal Logic.** Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2001

Example: Halpern and Shoham's Modal Logic of Time Intervals

Let $\mathbb{D} = \langle D, < \rangle$ be a linear order with domain D.

An interval over $\mathbb{D}$ is an ordered pair $[x, y]$, where $x, y \in \mathbb{D}$ and $x < y$.

There are 12 different binary ordering relations between two intervals:

relation	definition	example
after	$R_A([x,y],[w,z]) \quad = \quad =(y,w)$	
later	$R_L([x,y],[w,z]) \quad = \quad <(y,w)$	
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ends	$R_E([x,y],[w,z]) \quad = \quad <(x,w) \wedge =(y,z)$	
during	$R_D([x,y],[w,z]) \quad = \quad <(x,w) \wedge <(z,y)$	
overlaps	$R_O([x,y],[w,z]) \quad = \quad <(x,w) \wedge <(w,y) \wedge <(y,z)$	

and their inverse $R_{\overline{X}} = R_X^{-1}$ for each $X \in \{A, L, B, E, D, O\}$.

To each relation $R_{X \in \{A,\overline{A},L,\overline{L},B,\overline{B},E,\overline{E},D,\overline{D},O,\overline{O}\}}$ corresponds a modality $\langle X \rangle$.

Multi-modal logics

Definition

Given a non-empty set of *worlds* W, a *Kripke frame* is an object $F = \langle W, R_1 \dots R_n \rangle$ where each $R_i \subseteq W \times W$ is an *accessibility* relation.

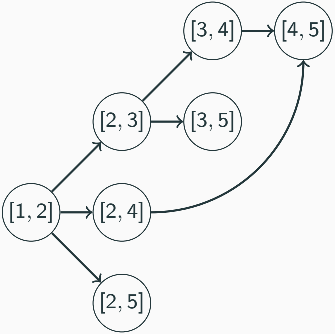

Figure 6: A Kripke frame for the relation R_A (*after*) of Halpern and Shoham's Modal Logic of Time Intervals; each world w_i represents an interval $[x_i, y_i]$.

Multi-modal logics

Definition

A *Kripke structure* (or *model*) is a Kripke frame enriched with a *valuation function* $V : W \to 2^{\mathcal{AP}}$, and it is denoted by $M = \langle F, V \rangle$.

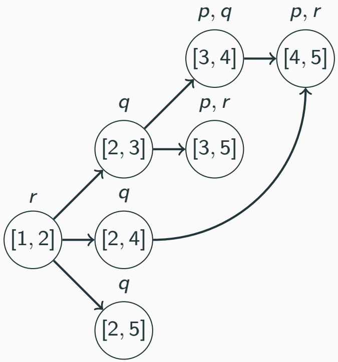

Figure 7: A Kripke structure for the Kripke frame in Fig. 6 and the set of propositional letters $\mathcal{AP} = \{p, q, r\}$; for each world, we represent only the propositional letters which are true in that world.

Multi-modal logics

Definition

Given a well-formed formula φ, we say that φ is *satisfied in M at w*, for some world w, and we denote it by $M, w \Vdash \varphi$, if and only if

$$\begin{array}{lll} M, w \Vdash p & \text{iff} & w \in V(p), \text{ for each } p \in \mathcal{A}P, \\ M, w \Vdash \neg\psi & \text{iff} & M, w \nVdash \psi, \\ M, w \Vdash \psi \vee \xi & \text{iff} & M, w \Vdash \psi \text{ or } M, w \Vdash \xi \\ M, w \Vdash \langle X_i \rangle \psi & \text{iff} & \text{there is } s \text{ s.t. } wR_i s \text{ and } M, s \Vdash \psi. \end{array}$$

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\end{array}
$$

Definition

A formula φ is *satisfiable* iff there exists a structure and a world in which it is satisfied, and *valid* if it is satisfied at every world in every structure.

Many-valued multi-modal logics

Definition

Let $\mathcal{A}P$ be a set of propositional letters and $\mathcal{A}$ a complete $\mathrm{FL_{ew}}$-algebra. The well-formed formulas of the *Multi-Modal Logic* $\mathrm{FL_{ew}}$-K_n are obtained by the following grammar:

$$\varphi ::= \alpha \mid p \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \to \psi \mid \langle X_i \rangle \varphi \mid [X_i]\varphi,$$

for $1 \leq i \leq n$, $p \in \mathcal{A}P$, and $\alpha \in \mathcal{A}$.

In $\mathrm{FL_{ew}}$-algebras, negation is typical defined as $\neg\varphi \equiv \varphi \to 0$.

However, the double negation axiom ($\neg\neg\varphi \equiv \varphi$) is **not** always vaild.

Hence, $\mathrm{FL_{ew}}$-K_n requires an explicit inclusion of all Boolean operators, as well as the universal version of every modality.

Many-valued multi-modal logics

Definition

Given a non-empty set of *worlds* W and a complete $\mathrm{FL_{ew}}$-algebra $\mathcal{A}$, an $\mathrm{FL_{ew}}$-*Kripke frame* is an object $\widetilde{F} = \langle W, \widetilde{R}_1 \ldots, \widetilde{R}_n \rangle$, where each $\widetilde{R}_i : (W \times W) \to \mathcal{A}$ is an *accessibility* function.

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Definition

An $\mathrm{FL}_{\mathrm{ew}}$-*Kripke structure* (or *model*) is an $\mathrm{FL}_{\mathrm{ew}}$-Kripke frame enriched with a *valuation function* $\widetilde{V} : (W \times \mathcal{A}P) \to \mathcal{A}$, and it is denoted by $\widetilde{M} = \langle \widetilde{F}, \widetilde{V} \rangle$.

Many-valued multi-modal logics

Definition

Given a well-formed formula φ, we compute its *value in* $\widetilde{M}$ *at* w, for some $w \in W$, by extending $\widetilde{V}$ to formulas, as follows:

$$\begin{array}{lcl}
\widetilde{V}(\alpha, w) & = & \alpha, \\
\widetilde{V}(\varphi \wedge \psi, w) & = & \widetilde{V}(\varphi, w) \cdot \widetilde{V}(\psi, w), \\
\widetilde{V}(\varphi \vee \psi, w) & = & \widetilde{V}(\varphi, w) \cup \widetilde{V}(\psi, w), \\
\widetilde{V}(\varphi \rightarrow \psi, w) & = & \widetilde{V}(\varphi, w) \hookrightarrow \widetilde{V}(\psi, w), \\
\widetilde{V}(\langle X_i \rangle \varphi, w) & = & \bigcup \{\widetilde{R}_i(w, s) \cdot \widetilde{V}(\varphi, s)\}, \\
\widetilde{V}([X_i]\varphi, w) & = & \bigcap \{\widetilde{R}_i(w, s) \hookrightarrow \widetilde{V}(\varphi, s)\}.
\end{array}$$

Many-valued multi-modal logics

Definition

A formula φ of $\mathrm{FL_{ew}}$-K$_n$ is α-*satisfied* at world w in an $\mathrm{FL_{ew}}$-Kripke structure $\widetilde{M}$ if and only if

$$\widetilde{V}(\varphi, w) \succeq \alpha.$$

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Definition

A formula φ is α-*satisfiable* if and only if there exists a structure and a world in which it is α-satisfied, and it is *satisfiable* if it is α-satisfiable for some $\alpha \in \mathcal{A}$, $\alpha \succ 0$; respectively, a formula is α-*valid* if it is α-satisfied at every world in every model, and *valid* if it is 1-valid.

Example: Halpern and Shoham's Modal Logic of Time Intervals

Definition

Let $\mathcal{A} = \langle A, \cap, \cup, \cdot, 0, 1\rangle$ a complete $\mathrm{FL}_{\mathrm{ew}}$-algebra.

An $\mathrm{FL}_{\mathrm{ew}}$-*linear order* is a structure of the type

$$\widetilde{\mathbb{D}} = \langle D, \widetilde{<}, \widetilde{=}\rangle,$$

where D is a *domain* enriched with two functions $\widetilde{<}, \widetilde{=} \colon D \times D \to A$, for which the following conditions apply for every $x, y, z \in D$:

$$\begin{aligned}
&\widetilde{=}(x,y) = 1 \text{ iff } x = y,\\
&\widetilde{=}(x,y) = \widetilde{=}(y,x),\\
&\widetilde{<}(x,x) = 0,\\
&\widetilde{<}(x,z) \succeq \widetilde{<}(x,y) \cdot \widetilde{<}(y,z),\\
&\text{if } \widetilde{<}(x,y) \succ 0 \text{ and } \widetilde{<}(y,z) \succ 0, \text{ then } \widetilde{<}(x,z) \succ 0,\\
&\text{if } \widetilde{<}(x,y) = 0 \text{ and } \widetilde{<}(y,x) = 0, \text{ then } \widetilde{=}(x,y) = 1,\\
&\text{if } \widetilde{=}(x,y) \succ 0, \text{ then } \widetilde{<}(x,y) \prec 1.
\end{aligned}$$

Example: Halpern and Shoham's Modal Logic of Time Intervals

Let $\widetilde{\mathbb{D}} = \langle D, \widetilde{<}, \widetilde{=} \rangle$ be a $\mathrm{FL_{ew}}$-linear order with domain D.

An interval over $\widetilde{\mathbb{D}}$ is an ordered pair $[x, y]$, where $x, y \in \mathbb{D}$ and $x \widetilde{<} y$.

There are 12 different binary ordering relations between two intervals:

relation	definition	example
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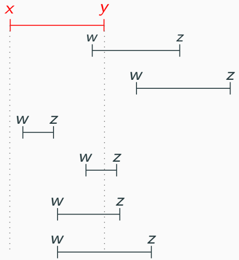

and their inverse $R_{\overline{X}} = R_X^{-1}$ for each $X \in \{A, L, B, E, D, O\}$.

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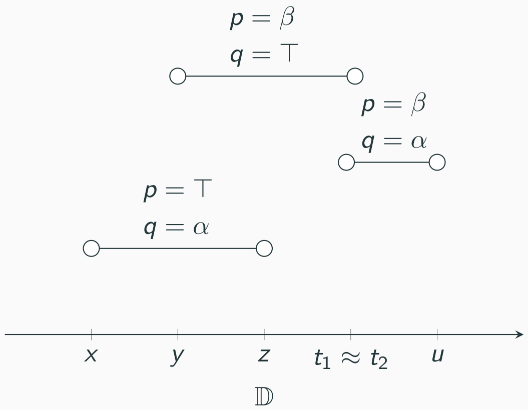

Figure 8: An interval model with six points and thirty intervals, where t_1 and t_2 are slightly apart. We consider an $\mathrm{FL_{ew}}$-algebra with 4 values $\bot \prec \alpha \prec \beta \prec \top$ and $\cap$ as t-norm, and that $R_O([y, t_1], [t_2, u]) = R_A([y, t_1], [t_2, u]) = \alpha$. In this model, $\langle O \rangle (p \wedge q) \succeq \alpha$ at interval I_2.

Reasoning in many-valued multi-modal logics

A tableau for many-valued multi-modal logics

During the last couple of years, we've been developing tableaux for α-satisfiability and α-validity in many-valued multi-modal logics[4].

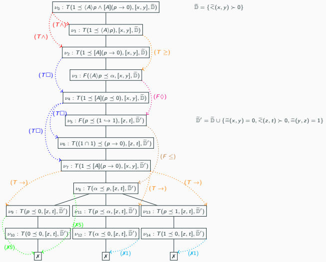

Figure 9: Fully evaluated Many-Valued Halpern and Shoam's interval temporal logic tableau for $\langle A \rangle p \wedge [A](p \rightarrow 0)$, $1 \in$ G3.

[4] Guillermo Badia et al. **"Fitting's Style Many-Valued Interval Temporal Logic Tableau System: Theory and Implementation".** In: *31st International Symposium on Temporal Representation and Reasoning (TIME 2024)*. 2024, 7:1–7:16

SoleReasoners.jl

An open-source implementation can be found in the SoleReasoners.jl package (`https://github.com/aclai-lab/SoleReasoners.jl`).

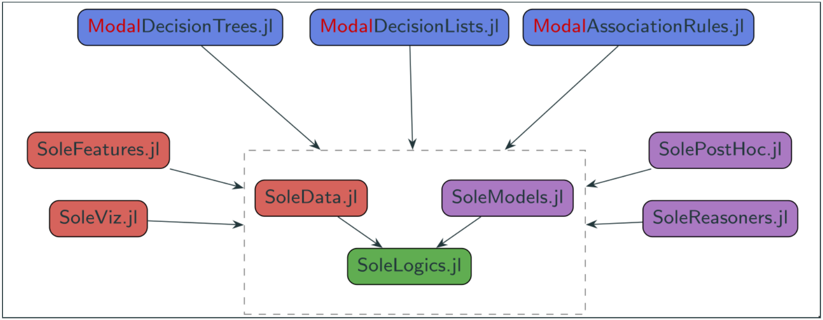

Figure 10: The SoLe ecosystem.

It is also part of the much larger SoLe framework, an open-source project written in Julia for Symbolic Learning, Reasoning and PostHoc.

The benchmarking problem

There is no available **benchmark** for many-valued multi-modal logics.

Hence, it is difficult not only to test the implementation **performance**, but also to assert its inherent **correctedness**.

In fact, at the moment, correctedness has been proved only for axioms (i.e., 1-validities), but that is not enough.

At the same time, attempts searching for modular formula structures of deterministic satisfiability/validity (as in [5][6] for the CRISP case) failed.

[5] Peter Balsiger, Alain Heuerding, and Stefan Schwendimann. **"A Benchmark Method for the Propositional Modal Logics K, KT, S4".** In: *Journal of Automated Reasoning* 24 (Jan. 2000)

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Solution: having more than one reasoner and comparing the results.

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Modal $\mathrm{FL_{ew}}$-algebra satisfiability through first-order translation

Modal $\mathrm{FL}_{\mathrm{ew}}$-algebra satisfiability through first-order translation

We define a function τ that translates the α-**satisfiability problem** for a modal FL$_{ew}$-algebra formula φ to a **two-sorted first-order logic**:

- one sort for the many-valued linear order
- one sort for the values of the algebra

Modal $\mathrm{FL_{ew}}$-algebra satisfiability through first-order translation

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- one sort for the many-valued linear order
- one sort for the values of the algebra

In the following slides:

- $||\varphi||_w \succeq \alpha$ should be read as φ at world w is at least α
- $p \in \mathcal{AP}$ is a propositional letter
- $\alpha, \beta \in \mathcal{A}$ are values from a specified $\mathrm{FL_{ew}}$-algebra
- φ, ψ are well-formed $\mathrm{FL_{ew}}$-K_n formulas

Modal $\mathrm{FL_{ew}}$-algebra satisfiability through first-order translation

For each $p \in \mathcal{AP}$, there is a function $\mathcal{P}(w)$ that returns a value from the algebra corresponding to the value of p at the world $w \in \mathcal{W}$.

Definition

$\tau(||p||_w \succeq \alpha) = \mathcal{P}(w) \succeq \alpha$

$\tau(||p||_w \preceq \alpha) = \mathcal{P}(w) \preceq \alpha$

$\tau(||\beta||_w \succeq \alpha) = \beta \succeq \alpha$

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Modal $\mathrm{FL_{ew}}$-algebra satisfiability through first-order translation

Definition

$\tau(||\varphi \wedge \psi||_w \succeq \alpha) = \exists x, y(\mathcal{A}(x) \wedge \mathcal{A}(y) \wedge \tau(||\varphi||_w \succeq x) \wedge \tau(||\psi||_w \succeq y) \wedge (x \cdot y \succeq \alpha))$

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$\tau(||\varphi \rightarrow \psi||_w \preceq \alpha) = \exists x, y(\mathcal{A}(x) \wedge \mathcal{A}(y) \wedge \tau(||\varphi||_w \succeq x) \wedge \tau(||\psi||_w \preceq y) \wedge (x \hookrightarrow y \preceq \alpha))$

Modal $\mathrm{FL_{ew}}$-algebra satisfiability through first-order translation

For each **accessibility relation** R_i in the considered $\mathrm{FL_{ew}}$-K_n logic, there is a function $\mathcal{R}_i(w, s)$ that returns a value from the algebra corresponding to the value of the accessibility relation between worlds $w, s \in \mathcal{W}$.

Definition

$$\tau(||\langle R_i\rangle\varphi||_w \succeq \alpha) = \exists x(\mathcal{A}(x) \wedge (x \succeq \alpha) \wedge \forall y(\mathcal{A}(y) \rightarrow ((y \succeq x) \leftrightarrow \forall z, s(\mathcal{A}(z) \wedge \mathcal{W}(s) \wedge \tau(||\varphi||_s \succeq z) \rightarrow \mathcal{R}_i(w, s) \cdot z \preceq y))))$$

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Implementation

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The translation has been implemented in the **Julia** language, and it's available open-source: `https://github.com/aclai-lab/LATD2025b`.

The idea is to translate the α-**satisfiability problem** for a given $\mathrm{FL}_{\mathrm{ew}}$-K$_n$ logics to first order logic and leverage a state of the art **sat/smt solver**.

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At the moment, it only supports:

- finite $\mathrm{FL_{ew}}$-algebras
- **Halpern and Shoham's Modal Logic of Time Intervals**
- the **z3** smt-solver

However, the output of the main process of the translation is an **.smt** file compliant to the **smt-lib** format, a valid input for most smt-solvers.

In the near future, it will also be part of the **SoleReasoners.jl** package.

Implementation

The implementation works in the following way:

1. Given an $\mathrm{FL_{ew}}$-algebra [7], declare a **sort** A for it, a distinct constant $a_1, \ldots, a_n \in A$ for each element in the algebra, and 4 functions *join*, *meet*, *monoid* and *implication* explicitly

[7] For a list of available $\mathrm{FL_{ew}}$-algebras, see the **ManyValuedLogics** submodule of **SoleLogics.jl**, `https://github.com/aclai-lab/SoleLogics.jl`)

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3. Recursively construct a string compliant to the **smt-lib** format applying the translation rules
4. Call an **smt-solver** (e.g., z3) on the produced file

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Tableau vs translation performance

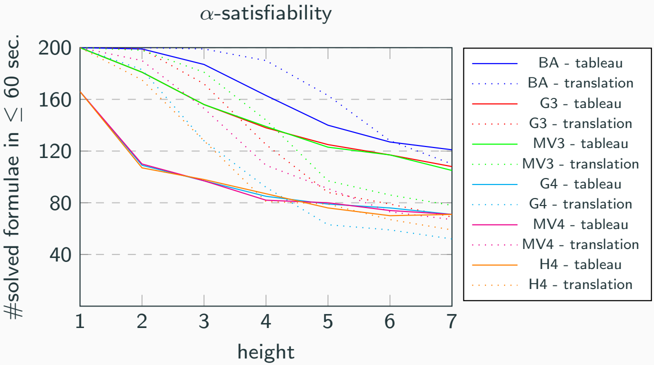

Figure 11: Tableau vs translation performance for solving α-satisfiability for **many-valued Halpern and Shoam's interval temporal logic**: how many formulae can be computed within a 60-second timeout over 1400 formulae (200 for each eight from 1 to 7) for algebras BA, G3, MV3, G4, MV4, H4.

Open-source advocacy

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Results: the two reasoning systems disagree on **16 formulas** out of 7800.

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Open-source doesn't only mean available for free.

Open-source means that everybody can **contribute**!

Please, **try** our packages, **test** them, **break** them, open an **issue** if you find any **bugs**, and feel free to **contribute** opening a **pull request**!

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Conclusions and Future work

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Today:

- introduced a **framework for many-valued multi-modal logics**
- introduced a translation for the α-**satisfiability problem** for many-valued multi-modal logics to a two-sorted first-order logic
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Today:

- introduced a **framework for many-valued multi-modal logics**
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- introduced an implementation for the translation for many-valued **Halpern and Shoam's modal logic of time intervals**

In the future:

- **support** for other **many-valued multi-modal logics**
- **support** for other **sat/smt solvers**
- further **testing** of **many-valued multi-modal reasoners**

Thank you for the attention!

Questions?