Generating and counting finite FL_{ew}-chains

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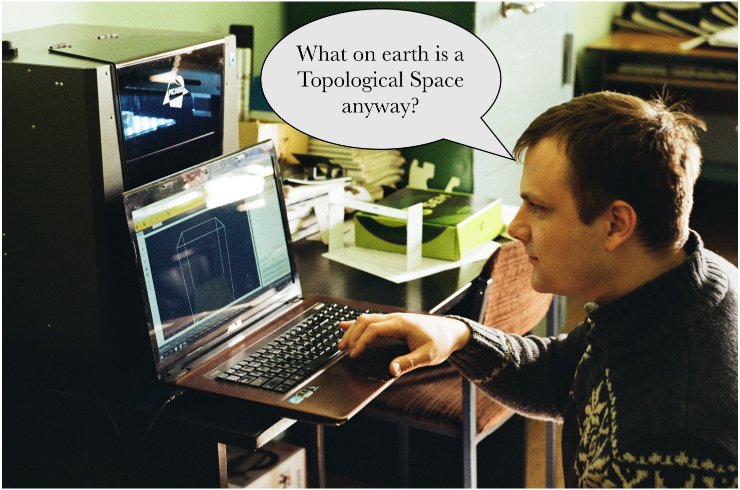
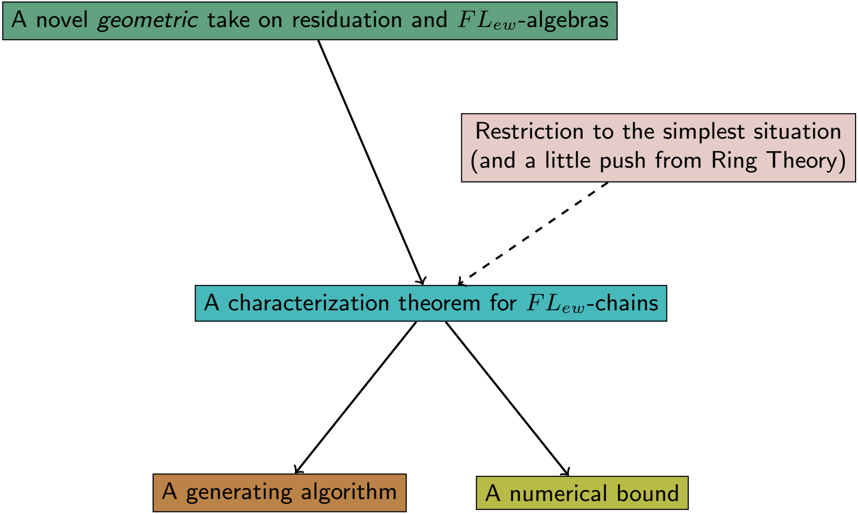
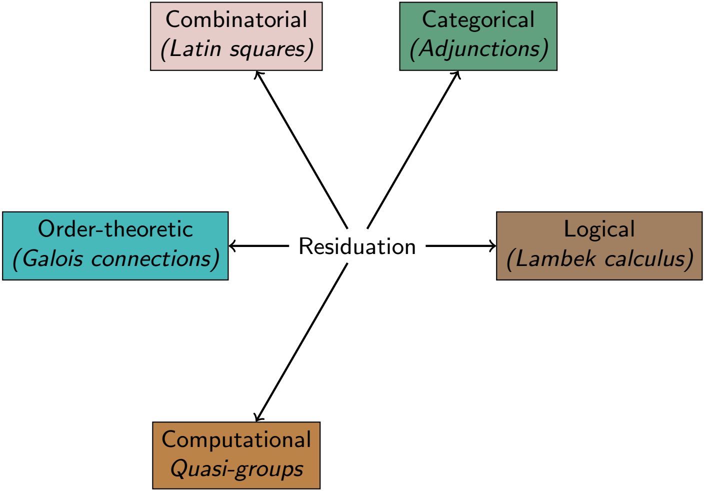


Today's plan



A novel *geometric* take on residuation and FL_{ew}-algebras
Restriction to the simplest situation (and a little push from Ring Theory)
A characterization theorem for FL_{ew}-chains
A generating algorithm
A numerical bound

Residuated maps form the bulk of much of order theory.
There are several equivalent but very diverse takes on residuation.



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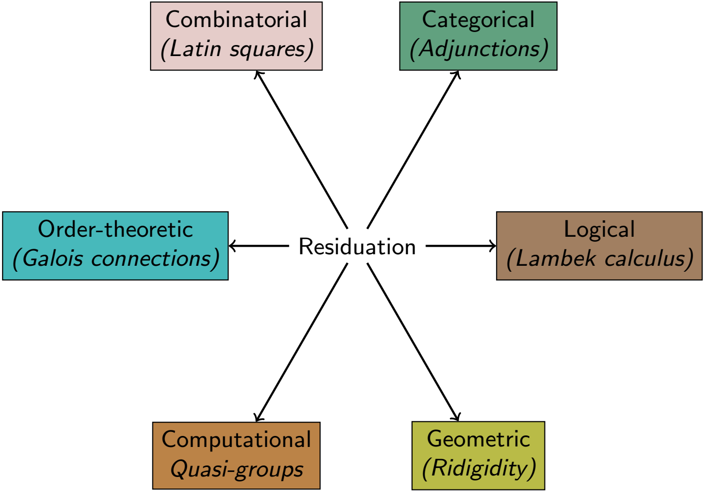

We propose a new one.

Topological setup

Definition

Let $(P, \leq)$ *be a poset. A* **lowerset** *is a subset* $S \subseteq P$ *such that*

$$\forall x \in P, \text{if } x \leq s \text{ for some } s \in S, \text{ then } x \in S.$$

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The principal lowersets are a basis for this topology.

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Proposition

A map $f \colon P \to Q$ between posets is **isotone** *(equiv. monotone) if and only if the preimage of a lowerset is again a lowerset.*
In other words, if and only if it is continuous with respect to the lower topology.

Geometric rigidity

Guiding principle

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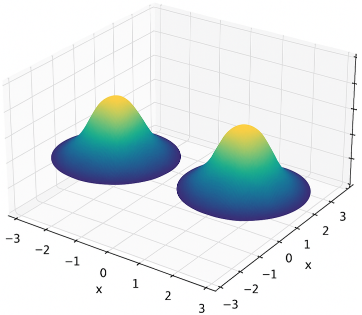
- $\mathcal{C}^\infty(D) = \{f\colon D \to \mathbb{R} \mid f \text{ is smooth}\}$ is a commutative ring with zero divisors;



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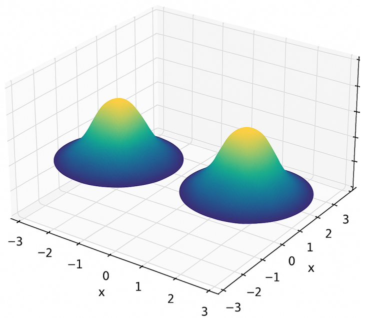
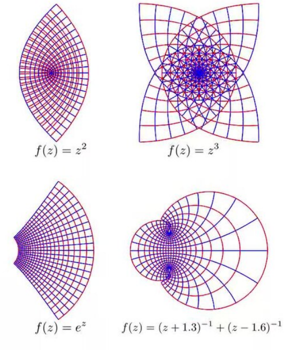
- $\mathcal{C}^\infty(D) = \{f : D \to \mathbb{R} \mid f \text{ is smooth}\}$ is a commutative ring with zero divisors;
- $\mathcal{O}(D) = \{f : D \to \mathbb{C} \mid f \text{ is holomorphic}\}$ is an integral domain.





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Naturally, the latter notion depends on the choice of bases.

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Intuitively, basic continuous maps are a **more rigid** form of continuous maps. This is being studied in greater generality as well.
Hence, they have better algebraic properties.

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This can also be seen through the lens of **monoidal categories** and **monoid objects**. (A topic for another day!)

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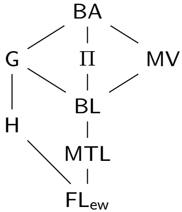

FL_{ew}-algebras are very important, as they comprise many (if not all) of the approaches to fuzzy logic:

- Gödel Logic, Heyting Algebras and Intuitionistic Logic;
- Product Logic;
- Chang's MV-algebras and Łukasiewicz Logic;
- t-norm Logics, BL Logic, MTL Logic.

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Boolean Ring	Boolean Algebra
Ideal	Order Ideal, Lowerset
Principal Ideal	Principal lowerset
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Given *our* definition of residuation, the best place to start is from PLPs.

A note from Algebraic Geometry

In Algebraic Geometry/Commutative Algebra, we know we can endow each Affine Algebraic Variety/Commutative Ring Spectrum with the **Zariski Topology**.

A basis for the Zariski Topology is given by **principal open sets**: complements of zeroes of one single polynomial.

$$D_f := \{(x_1, \ldots, x_n) \in \mathbb{A}^n(k) \mid f(x_1, \ldots, x_n) \neq 0\}.$$

A space in which every Zariski open set is principal is the **affine line** $\mathbb{A}^1(k)$, which is the simplest affine variety to begin with.
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Observation

In other words, PLPs can also be seen as the order-theoretic incarnation of **straight lines in geometry**.

Finite chains

But what actually *is* a PLP?

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We begin our study of FL_{ew} algebras by restricting ourselves to **finite chains**.

Observation

$$\begin{tikzcd} \textit{Residuated} \arrow[rr, Rightarrow] & & \textit{Isotone} \arrow[ll, Rightarrow, bend left, "\textit{Over a PLP}"] \end{tikzcd}$$

With this, we obtain the following result:

[1] More classification of PLPs is coming!

Finite FL_{ew}-chains

The lattice structure is predetermined, so we only have to find the multiplication.

Theorem (Characterization of finite FL_{ew}-chains)

The quintuple $(C_n, \leq, 0, 1, \cdot)$ is a FL_{ew}-algebra if and only if $(C_n, \cdot)$ is an associative magma and in its Cayley table:

- *The first row (and column) consists only of zeros;*
- *The last column, read from top to bottom, consists of all the elements $0, 1, 2, \ldots, n-1$, in this order;*
- *Every row and every column is weakly increasing;*
- *The table is symmetric with respect to the main diagonal.*

$\cdot$	**0**	**1**	**2**	**3**	**4**	**5**	**6**
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1
2	0	0	2	2	2	2	2
3	0	0	2	2	3	3	3
4	0	0	2	3	4	4	4
5	0	0	2	3	4	4	5
6	0	1	2	3	4	5	6

Implementation

An open-source implementation can be found in the ManyValuedLogics submodule of the SoleLogics.jl package (`https://github.com/aclai-lab/SoleLogics.jl`), a Julia package for working with propositional, multi-modal and many-valued logics.

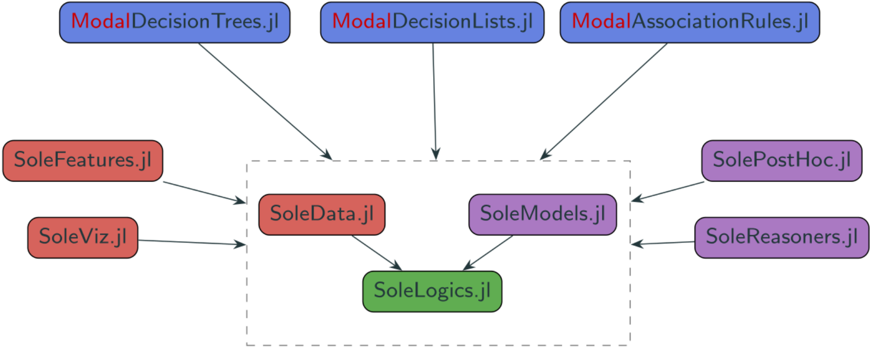

Figure: SoLe ecosystem.

It is also part of the much larger SoLe framework, an open-source project written in Julia for Symbolic Learning and Reasoning leveraging multi-modal and many-valued logics.

FiniteFLewChain data structure:

- parametrized over the number n of elements
- characterized by the Cayley table representing the t-norm operation
- each value of the chain is represented as Int8 $\{0, 1, 2, \ldots, n-1\}$
- we only need to represent $\binom{n-1}{2}$ elements

Algorithm Generate all weakly increasing sequences.

```
procedure WeaklyIncrRec(seqs, seq, min, max, l)
    if l = 0 then
        Push(seqs, seq)                          ▷ Ensure that seq is deep copied
    else
        for i ← min, max do
            Push(seq, i)
            WeaklyIncrRec(seqs, seq, i, max, l − 1)
            Pop(seq)
        end for
        return seqs
    end if
end procedure

procedure WeaklyIncr(min, max, l)
    return WeaklyIncrRec([], [], min, max, l)
end procedure
```

We employed Julia multithreading with shared memory, using a lock when pushing the newly generated Cayley table to the collection of all tables to prevent data races.

Algorithm Generate all FL_{ew}-chains with n elements.

procedure $\text{GenFL}_{ew}\text{ChRec}(cts, ct, min, max, l, n)$
    $wiseqs \leftarrow \text{WeaklyIncr}(min, max, l)$
    if $l = 1$ **then**
        for all $wiseq \in wiseqs$ **do**
            $ct \leftarrow \text{Concatenate}(ct, wiseq)$
            if $\text{CheckAssociativity}(ct)$ **then**
                $\text{Push}(cts, ct)$
            end if
        end for
    else
        for all $wiseq \in wiseqs$ **do**
            if $\text{isempty}(ct)$ **or** $\text{IsWIByCol}(wiseq, ct)$ **then**
                $ct \leftarrow \text{Concatenate}(ct, wiseq)$
                $\text{GenFL}_{ew}\text{ChRec}(cts, ct, wiseq[2], max + 1, l - 1, n)$
            end if
        end for
    end if
    return cts
end procedure

procedure $\text{GenFL}_{ew}\text{Ch}(n)$
    return $\text{GenFL}_{ew}\text{ChRec}([], [], 0, 1, n - 2, n)$
end procedure

Results

Usable and open-source tool that can be run on any common machine:

- Generating all finite FL$_{ew}$-chains for a given number of elements $n \leq 9$ only requires a few seconds on a single core execution
- Generating all finite FL$_{ew}$-chains with $n = 10$ in less than 10 minutes with a multithreaded execution employing 4 cores (8 threads) on an i5-8250u CPU

elements	chains
1	1
2	1
3	2
4	6
5	22
6	94
7	451
8	2386
9	13775
10	86417
11	590489

Table: Number of generated finite FL$_{ew}$-chains up to 11 elements

Also, this sequence is A030453 in the OEIS.

We recall two combinatorial results.

[2] We thank professor Max Alekseyev from the George Washington University for his help with this proof.

Numerical estimates

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Proposition ([2])

Let $\mathrm{WI}(a, b; \ell)$ *be the set of all weakly increasing sequences of length* ℓ *in the range* $[a; b]$ *and* $N \in \mathbb{N}$. *Then:*

$$\operatorname{card} \mathrm{WI}(a, b; \ell) = \binom{(b-a)+\ell}{\ell} \qquad \prod_{j=0}^{N} \binom{N}{j} = \prod_{k=1}^{N} k^{2k-N-1}.$$

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With them, we can prove:

Theorem (Numerical estimate for the number of finite FL_{ew}-chains)

Let $n \in \mathbb{N}$; *then the number of* FL_{ew}*-chains with* n *elements is at most*

$$b(n) = \prod_{k=1}^{n-1} k^{2k-n}.$$

This is sequence A001142 on the OEIS.

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- The *Sole* framework grows every day.